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JAN 78 C OPRIAN, V TANEJA, D VOSS, L A AROIAN N00014-77-C-0438

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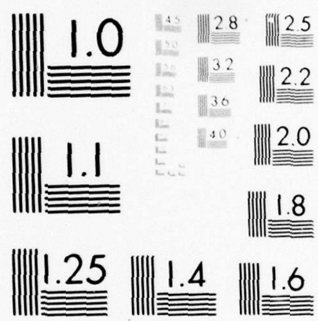

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INTERRELATIONSHIPS BETWEEN AUTOREGRESSIVE AND MOVING AVERAGE MODELS--THE ARMA MODEL: GENERAL CONSIDERATIONS IN M DIMENSIONS

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AES-7804 ERRATA SHEET

INTERRELATIONSHIPS BETWEEN AUTOREGRESSIVE AND MOVING AVERAGE MODELS--THE ARMA MODEL: GENERAL CONSIDERATIONS IN M DIMENSIONS

Page 1, line 3 from bottom: should read $\rho_{n,k} = \ldots$

Page 1, bottom line: should read $E(\tilde{z}^2_{x,t}) < \infty$ and...

Page 2, line 3 from bottom: should read "where $F^n_x = \ldots$

Page 3, line 8 from bottom, insert comma between B_{x_1} and B_t.

Page 3, line 4 from bottom, insert comma between B_x and B_t.

Page 4, line 11 from bottom: should read "converge on $\overset{m}{\underset{i=-m}{x}} S_i$,"

Page 4, line 10 from bottom, add "$S_i = \{F_{x_i} : |F_{x_i}| \leq 1\}$.

Page 5, line 4 from top: should read $E[\tilde{z}_{x-\ell,t-\kappa}, a_{x,t}]$

Page 5, Eq.(2.3) should read $\gamma_{00} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k}\gamma_{n,-k} \ldots$

Page 6, Eq.(3.1), remove parenthesis from exponent $e^{-2\pi if}$ and place below.

Page 6, lines 3 and 4: $e^{-i2\pi\sigma}$ should read $e^{-i2\pi f}$.

Page 7, line 4, beginning, should read $\Phi(B_x, B_t)$.

Page 7, line 6, should read $\phi_{01} = \phi_1$

Page 7, line 2 from bottom, add the term $+\gamma_{z,a}(\ell,\kappa)$ at end of line.

Page 8, line 9 from bottom, should read $\gamma_{01} = \phi_2\gamma_{-10} \ldots$

Page 8, line 3 from bottom, should read $\gamma_{z,a}(00) = \sigma^2_a$

Page 9, line 4 from top should read

$$\frac{\gamma_{01} - \phi_2\gamma_{-10} - \phi_1\gamma_{00}}{\sigma^2_a}$$

Page 9, line 11 from bottom: should read $\gamma_{\ell,\kappa}$.

Page 9, line 5 from bottom. Replace "Taneja et al." by Voss et al. Add the following: If $\theta_1 = \theta_2 = 0$, the ARMA model reduces to the 1 dim autoregressive model of temporal and spatial order 1. The equations for the autocovariance then agree with the results obtained by Taneja et al.

(over)

Page 10, line 5 from bottom, last term of equation on that line is $z_{x,t-2}$.

Page 11, lines 3 & 4 from bottom, insert "1" at end of each line.

Page 11, bottom line, equations should read:

$$\left.\begin{array}{l} |\theta_4 + \theta_3| < 1 \\ |\theta_4 - \theta_3| < 1 \end{array}\right. \Longleftrightarrow |\theta_3| + |\theta_4| \le 1$$

Page 12, line 8 from bottom, last term in equation is σ_a^2.

Page 14, line 4 from top, last section should read

$\{(\phi_4-\theta_4) + \phi_1(\phi_1-\theta_1)\}\sigma_a^2$

Page 14, line 5 from top should read:

$\gamma_{za}(0\text{-}1) = (\phi_1-\theta_1)\sigma_a^2 \quad \gamma_{za}(-1\text{-}2) = \{\phi_1(\phi_2-\theta_2) + \phi_2(\phi_1-\theta_1) + (\phi_3-\theta_3)\}\sigma_a^2$

Page 14, line 6 from top should read:

$\gamma_{za}(-1\text{-}1) = (\phi_2-\theta_2)\sigma_a^2$

Page 14, line 10 from bottom should read:

$\gamma_{01} = \phi_1\gamma_{00}+\phi_2\gamma_{-10}+\phi_3\gamma_{-1-1}+\phi_4\gamma_{0-1}-\theta_1\sigma_a^2-\theta_3(\phi_2-\theta_2)\sigma_a^2-\theta_4(\phi_1-\theta_1)\sigma_a^2$

Page 14, line 9 from bottom should read:

$\gamma_{10} = \phi_1\gamma_{1-1}+\phi_2\gamma_{0-1}+\phi_3\gamma_{0-2}+\gamma_4\gamma_{1-2}-\theta_2(\phi_1-\theta_1)\sigma_a^2-\theta_3\{(\phi_4-\theta_4)+\phi_1(\phi_1-\theta_1)\}\sigma_a^2$

Page 14, line 8 from bottom should read:

$\gamma_{11} = \phi_1\gamma_{10}+\phi_2\gamma_{00}+\phi_3\gamma_{0-1}+\phi_4\gamma_{1-1}-\theta_2\sigma_a^2-\theta_3(\phi_1-\theta_1)\sigma_a^2$

Page 17, lines 7, 6, 5, 4 from bottom, $\le$ should be replaced by $<$.

Page 18, lines 4, 5, 6, 7 from top, $\le$ should be replaced by $<$.

Page 19, line 7 from top, last term should read (000)

Page 21, line 9 from bottom should read $p(g,f) = p(g_1,g_2,f)$...

Page 21, line 8 from bottom should read $p(g_1,g_2,f)$.....

References, second page, add: Taneja, V.A., Aroian, L.A. (1977). Time series in m dimensions, autoregressive models.

AES-7804

INTERRELATIONSHIPS BETWEEN AUTOREGRESSIVE
AND
MOVING AVERAGE MODELS - THE ARMA MODEL:
GENERAL CONSIDERATIONS IN M DIMENSIONS

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1. INTRODUCTION.

We describe a general linear stochastic model which supposes a time series to be generated by a linear aggregation of random shocks at various temporal and spatial locations. Letting $x = (x_1, x_2, \ldots, x_m)$, a M dimensional vector, the general autoregressive-moving average model (ARMA) of M-dimensional time series is:

$$(1.1) \quad \tilde{z}_{x,t} = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \left[\phi_{n,k} \tilde{z}_{x+n,t-k} - \Theta_{n,k} a_{x+n,t-k}\right] + a_{x,t},$$

where $n = (n_1, n_2, \ldots, n_m)$ and $\sum\limits_{n=-\infty}^{\infty}$ denotes the M-dimension sum over each of the M components of n (i.e. $\sum\limits_{n=-\infty}^{\infty} = \sum\limits_{n_1=-\infty}^{\infty} \sum\limits_{n_2=-\infty}^{\infty} \cdots \sum\limits_{n_m=-\infty}^{\infty}$) and $\tilde{z}_{x,t} = z_{x,t} - E(z_{x,t})$, the deviation from the mean.

The $a_{x,t}$ are independent random shocks, so that their autocovariance function is:

$$\gamma_{n,k} = E(a_{x,t} a_{x+n,t-k}) = \begin{cases} \sigma_a^2, & n=o, k=o \text{ i.e } n = (o,o,\ldots,o) \\ o, & \text{otherwise} \end{cases}$$

and their autocorrelation function is:

$$\phi_{n,k} = \begin{cases} 1, & n=o, k=o \\ o, & \text{otherwise} \end{cases}$$

We also assume that $\tilde{z}_{x,t}$ is a weakly stationary process, i.e.

$$E(\tilde{z}_{n,t}) < \infty \quad \text{and} \quad E(\tilde{z}_{x,t_1} \tilde{z}_{y,t_2}) = \sigma_z^2 \phi_{|x-y|,|t_1-t_2|}$$

In this paper we explicitly focus our attention on the special case of (1.1) in which only a finite number of the coefficients are non-zero, that is:

$$(1.2) \quad \tilde{z}_{x,t} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} \tilde{z}_{x+n,t-k} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k} a_{x+n,t-k} + a_{x,t}$$

Whether or not the coefficients are zero it is easy to represent the process (1.2) in terms of shift operators.

$$B_t \tilde{z}_{x,t} = \tilde{z}_{x,t-1}$$

$$B_{x_i} \tilde{z}_{x,t} = \tilde{z}_{x-\delta_i,t} \text{ where } \delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im}) \text{ and } \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & \text{otherwise} \end{cases}$$

$$F_{x_i} \tilde{z}_{x,t} = \tilde{z}_{x+\delta_i,t}$$

Powers of these operators are defined in the usual manner, for example:

$$B^2_{x_i} \tilde{z}_{x,t} = B_{x_i}(B_{x_i} \tilde{z}_{x,t}) = \tilde{z}_{x-2\delta_i,t}$$

In addition, we note that B_{x_i} is the inverse of F_{x_i}, i.e. $B^{-1}_{x_i} = F_{x_i}$

Equation (1.2) can be written in terms of these shift operators:

$$(1.3) \quad \tilde{z}_{x,t} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} F^n_x B^k_t \tilde{z}_{x,t} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k} F^n_x B^k_t a_{x,t} + a_{x,t}$$

where $F^m_x = (F^{n_1}_{x_1}, F^{n_2}_{x_2}, \ldots, F^{n_m}_{x_m}) = (B^{-n_1}_{x_1}, B^{-n_2}_{x_2}, \ldots, B^{-n_m}_{x_m})$

Equation (1.3) can be rewritten in the form:

$$(1.4) \quad (1 - \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} F^n_x B^k_t) \tilde{z}_{x,t} = (1 - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k} F^n_x B^k_t) a_{x,t}$$

or $\Phi(B_x, B_t)\tilde{z}_{x,t} = \Theta(B_x, B_t)a_{x,t}$ where

$$\Phi(B_{x_1} B_t) = \Phi(B_{x_1}, B_{x_2}, \ldots, B_{x_m}, B_t) \text{ and}$$

$$\Theta(B_{x_1} B_t) = \Theta(B_{x_1}, B_{x_2}, \ldots, B_{x_m}, B_t)$$

We may consider this as an ARMA model of temporal orders r and s, and of spatial orders (p+q) and (u+v). By spatial order p+q we mean of order p_i+q_i in dimension i for each of the M dimensions of the autoregressive portion of the model. Similarly for the moving average portion of the model, the spatial order is u+v or u_i+v_i in each dimension i. We then refer to this as a M-dim ARMA model of order (r,s; p,q; u,v).

The ARMA model of equation (1.4) may be considered as a M-dim autoregressive process:

$$\Phi(B_{x_1} B_t)\tilde{z}_{x,t} = e_{x,t}, \text{ where } e_{x,t} = \Theta(B_x, B_t)a_{x,t}$$

is a M-dim moving average process.

This model may also be considered as a M-dim moving average process:

$$\tilde{z}_{x,t} = \Theta(B_x B_t)b_{x,t}, \text{ where } b_{x,t} \text{ follows the}$$

M-dim autoregressive process defined below:

$$\Phi(B_x, B_t)b_{x,t} = a_{x,t} \text{ so that}$$

$$\Phi(B_x, B_t)\tilde{z}_{x,t} = \Theta(B_x, B_t)\Phi(B_x, B_t)b_{x,t} = \Theta(B_x, B_t)a_{x,t}$$

The moving average terms on the right of equation (1.4) will not affect the conditions for stationarity of an autoregressive process in M-dimensions. Therefore, these restrictions on the parameters of the autoregressive model alone will apply to the combined ARMA model. The condition for stationarity is that the autocovariance generating function $\Phi(B_x, B_t)$ must converge for $|B_t| \leq 1$ and $|B_x| \leq 1$, we mean $|B_{x_i}| \leq 1$, for $i = 1,2,\ldots,M$.

Similarly the conditions for invertibility of the process that apply in the moving average model also apply in the ARMA model. The condition for invertibility is that $\Pi(B_x, B_t) = \Theta^{-1}(B_x, B_t)$ must converge on $S_0 x S_{-1} x S_{-2} x \ldots x S_{-m}$, where $S_{-i} = \left\{B_{x_i} : |B_{x_i}| \leq 1\right\}$ and $S_0 = \left\{B_i : |B_t| \leq 1\right\}$.

2. AUTOCOVARIANCE.

The autocovariance of the mixed ARMA process may be gotten by multiplying equation (1.3) by $\tilde{z}_{x-\ell,t-\kappa}$, where $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ and taking expectations.

Writing equation (1.3) as:

$$\tilde{z}_{x,t} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} F_x^n B_t^k \tilde{z}_{x,t} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k} F_x^n B_t^k a_{x,t} + a_{x,t}$$

$$= \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} \tilde{z}_{x+n,t-k} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k} a_{x+n,t-k} + a_{x,t}$$

Multiplying by $\tilde{z}_{x-\ell,t-\kappa}$:

$$\tilde{z}_{x-\ell,t-\kappa} \tilde{z}_{x,t} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k} \tilde{z}_{x+n,t-k} \tilde{z}_{x-\ell,t-\kappa} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \phi_{n,k} a_{x+n,t-k} \tilde{z}_{x-\ell,t-\kappa} + a_{x,t} \tilde{z}_{x-\ell,t-\kappa}$$

Taking expectations:

$$(2.1) \quad \gamma_{\ell,\kappa} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k}\, \gamma_{\ell+n,\kappa-k} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k}\, \gamma_{z,a}(\ell+n, \kappa-k) + \gamma_{z,a}(\ell,\kappa)$$

where $\gamma_{\ell+n,\kappa-k} = E(\tilde{z}_{x+n,t-k}\, \tilde{z}_{x-\ell,t-\kappa}$ and $\gamma_{z,a}(\ell,\kappa)$ is the cross covariance between z and a and is defined by: $\gamma_{z,a}(\ell,k) = E\tilde{z}_{x-\ell,t-}\,, a_{x,t}$

Since $\tilde{z}_{x-\ell,t-\kappa}$ depends only on shocks up to time $t-\kappa$, it follows that $\gamma_{za}(\ell,\kappa) = 0$, $\kappa > 0$. Since the $a_{x,t}$ are independent random shocks it also follows that $\gamma_{za}(\ell,\kappa) = 0$, $\ell > 0$.

Consequently:

$$(2.2) \quad \gamma_{\ell,\kappa} = \sum_{n=-p}^{q} \sum_{k=1}^{r} \phi_{n,k}\, \gamma_{\ell+n,\kappa-k}, \quad \text{for either} \quad \kappa \geq s+1 \quad \text{or}$$

$\ell_i \geq u_i+1$, for some dimension i, $i = 1,2,\ldots,M$.

The variance of this process is:

$$(2.3) \quad \gamma_{00} = \sum_{n=-p}^{q} \phi_{n,k}\, \gamma_{n,-k} - \sum_{n=-u}^{v} \sum_{k=1}^{s} \theta_{n,k}\, \gamma_{z,a}(n,-k) + \sigma_a^2$$

For a given ARMA model, this system of equations can be solved to determine the parameters $\phi_{n,k}$ and $\theta_{n,k}$ in terms of the autocovariances $\gamma_{n,\kappa}$.

3. <u>Spectrum</u>:

The spectrum of a mixed process in M-dimensions follows from the M-dimensional autoregressive model and the M-dimensional moving average model in a manner similar to the zero dimensional case of Box and Jenkins.

$$(3.1)\quad p(g,f) = \frac{2\sigma_a^2 \,|\,\Theta(e^{-2\pi g},e^{-i2\pi f})\,|^2}{|\Phi(e^{-i2\pi g},e^{-i2\pi f})|^2}, \qquad \begin{array}{l} 0 \le g_i \le \tfrac{1}{2}, \\ 0 \le f \le \tfrac{1}{2}, \end{array}$$

where $g = (g_1, g_2, \ldots g_M)$ and $i = \sqrt{-1}$ and:

$$\Theta(e^{-2\pi i g},e^{-i2\pi\sigma}) = \Theta(e^{-i2\pi g_1},e^{-i2\pi g_2},\ldots,e^{-i2\pi g_m},e^{-i2\pi f})$$

$$\Phi(e^{-i2\pi g},e^{-i2\pi\sigma}) = \Phi(e^{-i2\pi g},\ldots,e^{-i2\pi g_m},e^{-i2\pi f})$$

4. Partial Autocorrelation:

The mixed ARMA process may be written in the form $a_{x,t} = \Theta^{-1}(B_x,B_t)\Phi(B_x,B_t)\tilde{z}_{x,t}$, where $\Theta^{-1}(B_x,B_t)$ is an infinite series in $B_{x_1},B_{x_2},\ldots,B_{x_m},B_t$ as in the moving average M-dimensional model and therefore the partial autocorrelation is quite similar to that of the pure moving average process.

5. ARMA Model (1.1; 1,0; 1,0)

We consider the 1-dim mixed process of temporal order 1, and spatial order 1, for both the autoregressive and moving average portions.

From equation (1.3) we obtain the following model:

$$(5.1)\quad \tilde{z}_{x,t} = \sum_{n=-1}^{0}\sum_{k=1}^{1} \phi_{n,k} F_x^n B_t^k \tilde{z}_{x,t} - \sum_{n=-1}^{0}\sum_{k=1}^{1} \Theta_{n,k} F_x^n B_t^k a_{x,t} + a_{x,t}$$

$$= \phi_{-11}\tilde{z}_{x-1,t-1} + \phi_{01}\tilde{z}_{x,t-1} - \Theta_{-11} a_{x-1,t-1} - \Theta_{01} a_{x,t-1} + a_{x,t}$$

From equation (1.4), we get the following form:

$$(1-\phi_{01}B_t-\phi_{1-1}B_xB_t)\tilde{z}_{x,t} = (1-\Theta_{01}B_t-\Theta_{-11}B_xB_t)a_{x,t}$$

For this model then:

$$(B_x,B_t) = 1-\phi_{01}B_t-\phi_{-11}B_xB_t, \text{ and } \Theta(B_x,B_t) = 1-\theta_{01}B_t-\theta_{-11}B_xB_t$$

For convenience let:

$$\phi_0 = \phi_1 \qquad \theta_0 = \theta_1$$
$$\phi_{-11} = \phi_2 \qquad \theta_{-11} = \theta_2$$

The model then may be written as:

$$\tilde{z}_{x,t} = \phi_2\tilde{z}_{x-1,t-1}+\phi_1\tilde{z}_{x,t-1}-\theta_2 a_{x-1,t-1}-\theta_1 a_{x,t-1}+a_{x,t}$$

From the corresponding first order 1-dimension auto-regressive model, this ARMA model is stationary if $|\phi_1+\phi_2| < 1$ and $|\phi_1-\phi_2| < 1$, or equivalently $|\phi_1| + |\phi_2| < 1$. From the corresponding 1-dim first order moving average model this ARMA model is invertible if $|\theta_1+\theta_2| < 1$ and $|\theta_1-\theta_2| < 1$, or equivalently $|\theta_1| + |\theta_2| < 1$.

The autocovariance function of this model may be obtained from equation (2.1):

$$(5.2)\quad \gamma_{\ell,\kappa} = \phi_{-11}\gamma_{\ell-1,\kappa-1}+\phi_{01}\gamma_{\ell,\kappa-1}-\theta_{-1,1}\gamma_{z,a}(\ell-1,\kappa-1)-\theta_{01}\gamma_{z,a}(\ell,\kappa-1)$$

$$= \phi_2\gamma_{\ell-1,\kappa-1}+\phi_1\gamma_{\ell,\kappa-1}-\Theta_2\gamma_{z,a}(\ell-1,\kappa-1)-\Theta_1\gamma_{z,a}(\ell,\kappa-1)+\gamma_{z,a}(\ell,\kappa)$$

The variance for this first model is:

$$(5.3)\quad \gamma_{00} = \phi_2\gamma_{-1\,1} + \phi_1\gamma_{0\,-1} - \theta_2\gamma_{za}(-1-1) - \theta_1\gamma_{z,a}(0-1) + \sigma_a^2$$

From equation (2.2), for $\ell \geq 2$, or $\kappa \geq 2$ the autocovariance depends only on the autoregressive coefficients ϕ_1 and ϕ_2. We therefore obtain the following:

$$(5.4)\quad \gamma_{\ell,\kappa} = \phi_{-1-1}\gamma_{-1-1} + \phi_{0-1}\gamma_{0-1} = \phi_2\gamma_{-1-1} + \phi_1\gamma_{0-1} \text{ for } \ell \geq 2, \text{ or } \kappa \geq 2.$$

The remaining autocovariance terms depend on both θ_1, θ_2 and ϕ_1, ϕ_2, and from equation (5.2) are given as:

$$(5.5)\quad \gamma_{01} = \phi_2\gamma_{-1} + \phi_1\gamma_{00} - \theta_2\gamma_{za}(-10) - \theta_1\gamma_{z,a}(00) + \gamma_{z,a}(01)$$

$$\gamma_{10} = \phi_2\gamma_{0-1} + \phi_1\gamma_{1-1} - \theta_2\gamma_{za}(0-1) - \theta_1\gamma_{z,a}(1-1) + \gamma_{z,a}(10)$$

$$\gamma_{11} = \phi_2\gamma_{00} + \phi_1\gamma_{10} - \theta_2\gamma_{za}(00) - \theta_1\gamma_{z,a}(10) + \gamma_{za}(11)$$

Since this ARMA model is of temporal order 1 and spatial order 1 for both the autoregressive and moving average portions of the model, the only non-zero cross covariance terms are:

$$\gamma_{z,a}(00) = \sigma_z^2$$

$$\gamma_{z,a}(0-1) = (\phi_1 - \theta_1)\,\sigma_a^2$$

$$\gamma_{z,a}(-1-1) = (\phi_2 - \theta_2)\,\sigma_a^2$$

Substituting these cross-variance terms in (5.3) and (5.5) yields the following:

$$\gamma_{00} = \sigma_a^2 + \phi_1\gamma_{0-1} + \phi_2\gamma_{-1-1} - \theta_1(\phi_1-\theta_1)\sigma_a^2 - \theta_2\ (\phi_2-\theta_2)\sigma_a^2$$

$$\gamma_{01} = \phi_2\gamma_{-10} + \phi_1\gamma_{00} - \theta_1\sigma_a^2;\ \theta_1 \ = \frac{\gamma_{01} - \ _2\gamma_{-10} - \phi_1\gamma_{00}}{\sigma_a^2}$$

$$\gamma_{10} = \phi_2\gamma_{0-1} + \phi_1\gamma_{1-1} - \theta_2(\phi_1-\theta_1)\sigma_a^2$$

$$\gamma_{11} = \phi_2\gamma_{00} + \phi_1\gamma_{10} - \theta_2\sigma_a^2;\ \ \theta_2 = \frac{\gamma_{11} - \phi_2\gamma_{00} - \phi_1\gamma_{10}}{\sigma_a^2}$$

The expressions for θ_1 and θ_2 may be substituted in the equations for γ_{00} and γ_{10}, which results in a pair of quadratic equations in ϕ_1 and ϕ_2, that may be solved for ϕ_1 and ϕ_2 and subsequently for θ_1 and θ_2 in terms of the autocovariances $\gamma_{\ell,r}$.

If $\phi_1 = \phi_2 = 0$, the ARMA model reduces to the 1 dim moving average model of temporal and spatial order 1. The equations for the autovariances then agree with the results obtained by Taneja et al.

The spectrum of this mixed first order process is obtained from equation (3.1) as:

$$p(f,g) = \frac{2\sigma_a^2|\Theta(e^{-i2\pi f}, e^{-i2\pi g})|^2}{|\ (e^{-i2\pi f}, e^{-i2\pi g})|^2}$$

$$= \frac{2\sigma_a^2|1-\theta_1 e^{-i2\pi f} - \theta_2 e^{-i2\pi f} e^{-i2\pi g}|^2}{|1-\phi_1 e^{-i2\pi f} - \phi_2 e^{-i2\pi f} e^{-i2\pi g}|^2}\ ,\ \begin{array}{l} 0 \le f \le ½ \\ 0 \le g \le ½ \end{array}$$

As in the case of the autocovariances, if $\theta_1 = \theta_2 = 0$, the spectrum reduces to that of the 1-dim first order autoregressive model; and if $\phi_1 = \phi_2 = 0$, *the spectrum reduces* to that of the 1-dim first order moving average model.

6. <u>ARMA Model (2,2; 1,0; 1,0)</u>:

The next model considered is a 1 dim mixed process that is of second order in time for both the autoregressive and the moving average portions and of first order in the space dimension for both the AR and MA portions, i.e. an ARMA model of the form (2,2; 1,0; 1,0).

From equation (1.3) we obtain the following form for this model:

$$\tilde{z}_{x,t} = \sum_{n=-1}^{0} \sum_{k=1}^{2} \phi_{n,k} F_x^n B_t^k \tilde{z}_{x,t} - \sum_{n=-1}^{0} \sum_{k-1}^{2} \theta_{n,k} F_x^n B_t^k a_{x,t} + a_{x,t}$$

$$= \phi_{-1,1} \tilde{z}_{x-1,t-1} + \phi_{-12,} \tilde{z}_{x-1,t-2} + \phi_{01} \tilde{z}_{x,t-1} + \phi_{02} \tilde{z}_{x,t-z} \tag{6.1}$$

$$-\theta_{-11} a_{x-1,t-1} - \theta_{-12} a_{x-1,t-2} - \theta_{01} a_{x,t-1} - \theta_{02} a_{x,t-2} + a_{x,t}$$

From equation (1.4) we get the following form:

$$(1-\phi_{01}B_t - \phi_{-11} B_x B_t - \phi_{-12} B_x B_t^2 - \phi_{02} B_t^2) \tilde{z}_{x,t} =$$

$$(1-\theta_{01}B_t - \theta_{-11} B_x B_t - \theta_{-12} B_x B_t^2 - \theta_{02} B_2^2) a_{x,t}$$

For convenience, let:

$$\phi_{01} = \phi_1 \qquad \phi_{-11} = \phi_2 \qquad \phi_{-12} = \phi_3 \qquad \phi_{02} = \phi_4$$

$$\theta_{01} = \theta_1 \qquad \theta_{-11} = \theta_2 \qquad \theta_{-12} = \theta_3 \qquad \theta_{02} = \theta_4$$

The model then may be rewritten as:

$$\tilde{z}_{x,t} = \phi_1 \tilde{z}_{x,t-1} + \phi_2 \tilde{z}_{x-1,t-1} + \phi_3 \tilde{z}_{x-1,t-2} + \phi_4 \tilde{z}_{x,t-2} - \theta_1 a_{x-1,t-1}$$
$$-\theta_2 a_{x-1,t-1} - \theta_3 a_{x-1,t-2} - \theta_4 a_{x,t-2} + a_{x,t}$$

From the corresponding 1 dim AR model of temporal order 2, and spatial order 1, this mixed ARMA model is stationary if:

$$|\phi_1 + \phi_2 + \phi_3 + \phi_4| < 1 \qquad |\phi_1 - \phi_2 + \phi_3 - \phi_4| < 1$$

$$|\phi_1 + \phi_2 - \phi_3 - \phi_4| < 1 \qquad |\phi_1 - \phi_2 - \phi_3 + \phi_4| < 1$$

From the corresponding 1 dim MA model of temporal order 2, and spatial order 1, this mixed ARMA model is invertible if the roots of $\Theta(B_x, B_t) = (1-(\theta_1+\theta_2 B_x)B_t - (\theta_4+\theta_3 B_x)B_t^2$ lie outside the region $|B_t| \leq 1$ and $|B_x| \leq 1$, therefore:

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 < 1 \qquad \theta_1 - \theta_2 - \theta_3 + \theta_4 <$$

$$\theta_1 - \theta_2 + \theta_3 + \theta_4 < 1 \qquad \theta_1 + \theta_2 - \theta_3 + \theta_4 <$$

$$\left.\begin{matrix} |\theta_3 + \theta_3| < 1 \\ |\theta_4 - \theta_3| < 1 \end{matrix}\right. \Longleftrightarrow |\theta_3|\theta_4| < 1$$

The autocovariance function for this model may be obtained from equation (2.1):

$$(6.2)\quad \gamma_{\ell,\kappa} = \phi_{-11}\gamma_{\ell-1,\kappa-1} + \phi_{-12}\gamma_{\ell-1,\kappa-2} + \phi_{01}\gamma_{\ell,\kappa-1} + \phi_{02}\gamma_{\ell,\kappa-2} - \theta_{-11}\gamma_{za}(\ell-1,\kappa-1)$$

$$- \theta_{12}\gamma_{za}(\ell-1,\kappa-2) - \theta_{01}\gamma_{za}(\ell,\kappa-1) - \theta_{02}\gamma_{za}(\ell,\kappa-2) + \gamma_{za}(\ell,\kappa)$$

$$= \phi_1\gamma_{\ell,\kappa-1} + \phi_2\gamma_{\ell-1,\kappa-1} + \phi_3\gamma_{\ell-1,\kappa-2} + \phi_4\gamma_{\ell,\kappa-2} - \theta_1\gamma_{za}(\ell,\kappa-1)$$

$$- \theta_2\gamma_{za}(\ell-1,\kappa-1) - \theta_3\gamma_{za}(\ell-1,\kappa-2) - \theta_4\gamma_{za}(\ell,\kappa-2) + \gamma_{za}(\ell,\kappa)$$

The variance for this ARMA model of temporal order 2 and spatial order 1, is:

$$(6.3)\quad \gamma_{00} = \phi_1\gamma_{0-1} + \phi_2\gamma_{-1-1} + \phi_3\gamma_{-1-2} + \phi_4\gamma_{0-2} - \theta_1\gamma_{za}(0-1) - \theta_2\gamma_{za}(-1-1)$$

$$- \theta_3\gamma_{za}(-1-2) - \theta_4\gamma_{za}(0-2) + \gamma_a^2$$

From equation (2.2) the autocovariance function for $\ell \geq 2$ or $\kappa \geq 3$ only depends on the autoregressive coefficients $\phi_1, \phi_2, \phi_3, \phi_4$.

We then have:

$$(6.4)\quad \gamma_{\ell,\kappa} = \phi_{-11}\gamma_{\ell-1,\kappa-1} + \phi_{-12}\gamma_{\ell-1,\kappa-2} + \phi_{01}\gamma_{\ell,\kappa-1} + \phi_{02}\gamma_{\ell,\kappa-2}$$

$$= \phi_1\gamma_{\ell,\kappa-1} + \phi_2\gamma_{\ell-1,\kappa-1} + \phi_3\gamma_{\ell-1,\kappa-2} + \phi_4\gamma_{\ell,\kappa-2},$$

$$\text{for } \ell \geq 2 \text{ or } \kappa \geq 3$$

The remaining autocovariance terms depend on both the autoregressive and the moving average coefficients and are given as obtained from equation (6.2):

$$\gamma_{01} = \phi_1\gamma_{00} + \phi_2\gamma_{-10} + \phi_3\gamma_{-1-1} + \phi_4\gamma_{0-1} - \theta_1\gamma_{za}(0,0) - \theta_2\gamma_{za}(-1,0) - \theta_3\gamma_{za}(-1,-1) - \theta_4\gamma_{za}(0,-1) + \gamma_{za}(0,1) \tag{6.5}$$

$$\gamma_{10} = \phi_1\gamma_{1-1} + \phi_2\gamma_{0-1} + \phi_3\gamma_{0-2} + \phi_4\gamma_{1-2} - \theta_1\gamma_{za}(1,-1) - \theta_2\gamma_{za}(0,-1) - \theta_3\gamma_{za}(0,-2) - \theta_4\gamma_{za}(1,-2) + \gamma_{za}(1,0)$$

$$\gamma_{11} = \phi_1\gamma_{10} + \phi_2\gamma_{00} + \phi_3\gamma_{0-1} + \phi_4\gamma_{1-1} - \theta_1\gamma_{za}(1,0) - \theta_2\gamma_{za}(0,0) - \theta_3\gamma_{za}(0,-1) - \theta_4\gamma_{za}(1,-1) + \gamma_{za}(1,1)$$

$$\gamma_{-11} = \phi_1\gamma_{-10} + \phi_2\gamma_{-20} + \phi_3\gamma_{-2-1} + \phi_4\gamma_{-1-1} - \theta_1\gamma_{za}(-1,0) - \theta_2\gamma_{za}(-1,1) - \theta_3\gamma_{za}(-2,-1) - \theta_4\gamma_{za}(-1,-1) + \gamma_{za}(-1,1)$$

$$\gamma_{02} = \phi_1\gamma_{01} + \phi_2\gamma_{-11} + \phi_3\gamma_{-10} + \phi_4\gamma_{00} - \theta_1\gamma_{za}(-1,1) - \theta_2\gamma_{za}(-1,1) - \theta_3\gamma_{za}(-1,0) - \theta_4\gamma_{za}(0,0) + \gamma_{za}(0,2)$$

$$\gamma_{12} = \phi_1\gamma_{11} + \phi_2\gamma_{01} + \phi_3\gamma_{00} + \phi_4\gamma_{10} - \theta_1\gamma_{za}(1,1) - \theta_2\gamma_{za}(0,1) - \theta_3\gamma_{za}(0,0) - \theta_4\gamma_{za}(1,0) + \gamma_{za}(1,2)$$

Since this ARMA model is of temporal order 2, and spatial order 1 for both the autoregressive and moving average portions, the only nonzero cross covariance terms are:

$$\gamma_{za}(0\,0) = \sigma_a^2 \qquad \gamma_{za}(0\text{-}2) = (\phi_4-\theta_4) + \phi_1(\phi_1-\theta_1)\,\sigma_a^2$$

$$\gamma_{za}(0\text{-}1) = (\phi_1-\theta)\sigma_a^2 \qquad \gamma_{za}(\text{-}1\text{-}2) = \phi_1(\phi_2-\theta_2) + \phi_2(\phi_1-\theta_1) + (\phi_3-\theta_3)\,\sigma_a^2$$

$$\gamma_{za}(\text{-}1\text{-}1) = (\phi_1-\theta)\sigma_a^2$$

Substituting these cross covariance terms in equations (6.3) and (6.5) yields the following:

$$(6.6)\quad \gamma_{00} = \phi_1\gamma_{0-1} + \phi_2\gamma_{-1-1} + \phi_3\gamma_{-1-2} + \phi_4\gamma_{0-2} - \theta_1(\phi_1-\theta_1)\sigma_a - \theta_2(\phi_2-\theta_2)\sigma_a^2$$

$$- \theta_3\{\phi_1(\phi_2-\theta_2)+\phi_2(\phi_1-\theta_1)+(\phi_3-\theta_3)\}\sigma_a^2 - \theta_4\{(\phi_4-\theta_4)+\phi_1(\phi_1-\theta_1)\}\sigma_a^2+\sigma_a^2$$

$$\gamma_{01} = \phi_1\gamma_{00}+\phi_2\gamma_{-10}+\phi_3\gamma_{-1-1}+\phi_4\gamma_{0-1}-\theta_1\,{}^2_a - \theta_3\{\phi_2-\theta_2\}\sigma_a^2 - \theta_4\{\phi_1-\theta_1\}\sigma_a^2$$

$$\gamma_{10} = \phi_1\gamma_{1-1}+\phi_2\gamma_{0-1}+\phi_3\gamma_{0-2}+\phi_4\gamma_{1-2} - \theta_2\{\phi_1-\theta_1\}\sigma_a^2-\theta_3\{(\phi_4-\theta_4)+\phi_1(\phi_1-\theta_1)\}\sigma_a^2$$

$$\gamma_{11} = \phi_1\gamma_{10}+\phi_2\gamma_{00}+\phi_3\gamma_{0-1}+\phi_4\gamma_{1-1}-\theta_2\sigma_a^2-\theta_3\{\phi_1-\theta_1\}\sigma_a^2$$

$$\gamma_{-11} = \phi_1\gamma_{-10}+\phi_2\gamma_{-20}+\phi_3\gamma_{-2-1}+\phi_4\gamma_{-1-1}-\theta_4\{\phi_2-\theta_2\}\sigma_a^2$$

$$\gamma_{02} = \phi_1\gamma_{01}+\phi_2\gamma_{-11}+\phi_3\gamma_{-10}+\phi_4\gamma_{00}-\theta_4\sigma_a^2$$

$$\gamma_{12} = \phi_1\gamma_{11}+\phi_2\gamma_{01}+\phi_3\gamma_{00}+\phi_4\gamma_{10}-\theta_3\sigma_a^2$$

Through an iterative procedure this set of AR and MA coefficients may be solved in terms of the autocovariances. If $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, this ARMA model reduces to the 1-dim moving average model of temporal order 2 and spatial order 1.

The above equations for the autocovariance function agree with the results obtained by Voss et al. If the $\theta_1 = \theta_2 = \theta_3 = \theta_4$ are zero, the model reduces to the 1-dim autoregressive model of temporal order 2 and spatial order 1. These autocovariances agree with the results obtained by Taneja et al.

The spectrum of this mixed process of temporal order 2 and spatial order 1 is obtained from equation (3.1) as:

$$p(g,f) = \frac{2\,{}^{2}_{a}\ |\ \Theta(e^{-i2\pi g}, e^{-i2\pi f})\ |^2}{|\ \Phi(e^{-i2\pi g}, e^{-i2\pi f})\ |^2}$$

$$= \frac{2\sigma_a^2\ |\ 1-\Theta_2 e^{-i2\pi g} e^{-i2\pi f} - \Theta_3 e^{-i2\pi g} e^{-i2\pi f} - \Theta_1 e^{-i2\pi f} - \Theta_4 e^{-i4\pi f}\ |^2}{|\ 1-\phi_1 e^{-i2\pi f} - \phi_2 e^{-i2\pi f} e^{-i2\pi g} - \phi_3 e^{-i4\pi f} e^{-i2\pi g} - \phi_4 e^{-i4\pi f}\ |^2},$$

$$0 \leq f \leq \tfrac{1}{2}$$
$$0 \leq g \leq \tfrac{1}{2}$$

As in the case of the autocovariance function if $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, the spectrum of the mixed process reduces to that of the 1-dim moving average model of temporal order 2 and spatial order 1. If $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, the spectrum reduces to that of the 1-dim autoregressive model of temporal order 2 and spatial order 1.

7. ARMA Model (1,1; 1,0, 1,0; 1,0, 1,0):

The next model considered is a 2 dimensional mixed model of first order in time and each of the 2 space dimensions, for both the autoregressive and the moving average portions, i.e. and ARMA model of the form (1,1; 1,0, 1,0; 1.0,1,0).

x dim	y dim	x dim	y dim
AR	AR	MA	MA

From equation (1.3) we obtain the following form for the model:

$$(7.1)\quad \tilde{z}_{x,y,t} = \sum_{n_1=-1}^{0} \sum_{n_2=-1}^{0} \sum_{k=1}^{1} \phi_{n_1,n_2,k} F_x^{n_1} F_y^{n_2} B_t^k \tilde{z}_{x,y,t}$$

$$- \sum_{n_1=-1}^{0} \sum_{n_2=-1}^{0} \sum_{k=1}^{1} \theta_{n_1,n_2,k} F_x^{n_1} F_y^{n_2} B_t^k a_{x,y,t} + a_{x,y,t}$$

$$= \phi_{-1-11} \tilde{z}_{x-1,y-1,t-1} + \phi_{-101} \tilde{z}_{x-1,y,t-1} + \phi_{0-11} \tilde{z}_{x,y-1,t-1}$$

$$+ \phi_{001} \tilde{z}_{x,y,t-1} - \theta_{-1-11} a_{x-1,y-1,t-1} - \theta_{101} a_{x-1,y,t-1}$$

$$- \theta_{0-11} a_{x,y-1,t-1} - \theta_{001} a_{x,y,t-1} + a_{x,y,t}$$

From equation (1.4) we get the following form:

$$(1 - \phi_{001} B_t - \phi_{-101} B_x B_t - \phi_{0-11} B_y B_t - \phi_{-1-11} B_x B_y B_t)\tilde{z}_{x,y,t}$$

$$= (1 - \theta_{001} B_t - \theta_{-101} B_x B_t - \theta_{0-11} B_y B_t - \theta_{-1-11} B_x B_y B_t) a_{x,y,t}$$

For this model then:

$$\Phi(B_x B_y B_t) = 1 - (\phi_{001} - \phi_{-101} B_x - \phi_{0-11} B_y - \phi_{-1-11} B_x B_y) B_t$$

$$\Theta(B_{x_1} B_y B_t) = 1 - (\theta_{001} - \theta_{-101} B_x - \theta_{0-11} B_y - \theta_{-1-11} B_x B_y) B_t$$

For convenience, let:

$$\phi_{001} = \phi_1 \quad \phi_{-101} = \phi_2 \quad \phi_{0-11} = \phi_3 \quad \phi_{-1-11} = \phi_4$$

$$\theta_{001} = \theta_1 \quad \theta_{-101} = \theta_2 \quad \theta_{0-11} = \theta_3 \quad \theta_{-1-11} = \theta_4$$

The model then may be rewritten as:

$$\tilde{z}_{x,y,t} = \phi_1 \tilde{z}_{x,y,t-1} + \phi_2 \tilde{z}_{x-1,y,t-1} + \phi_3 \tilde{z}_{x,y-1,t-1} + \phi_4 \tilde{z}_{x-1,y-1,t-1}$$

$$- \theta_1 a_{x,y,t-1} - \theta_2 a_{x-1,y,t-1} - \theta_3 a_{x,y-1,t-1} - \theta_4 a_{x-1,y-1,t-1} + a_{x,y,t}$$

From the corresponding 2 dim autoregressive model of temporal order 1, and spatial order 1 in both x and y directions, this mixed model is stationary if:

$$|\phi_1 + \phi_2 + \phi_3 + \phi_4| \leq 1$$
$$|\phi_1 - \phi_2 + \phi_3 - \phi_4| \leq 1$$
$$|\phi_1 + \phi_2 - \phi_3 - \phi_4| \leq 1$$
$$|\phi_1 - \phi_2 - \phi_3 + \phi_4| \leq 1$$

From the corresponding 2 dim moving average model of temporal order 1, and spatial order 1 in x and y, this mixed model is invertible if $\Pi(B_x, B_y, B_t) = \Theta^{-1}(B_x, B_y, B_t)$ converges

on $S_0 \times S_{-1} \times S_{-2}$, where $S_0 = \left\{B_t : |B_t| \le 1\right\}$ and

$S_{-i} = \left\{B_{x_i} : |B_{x_i}| \le 1\right\}$. Since $\Pi(B_x, B_y, B_t) = \left[1-(\theta_1+\theta_2B_x+\theta_3B_y+\theta_4B_xB_y)B_t\right]^{-1}$,

this mixed model is invertible if:

$$|\theta_1 + \theta_2 + \theta_3 + \theta_4| \le 1$$
$$|\theta_1 - \theta_2 - \theta_3 + \theta_4| \le 1$$
$$|\theta_1 + \theta_2 - \theta_3 - \theta_4| \le 1$$
$$|\theta_1 - \theta_2 + \theta_3 - \theta_4| \le 1$$

The autocovariance function for this model may be obtained from equation (2.1):

$$(7.2)\quad \gamma_{\ell_1,\ell_2,\kappa} = \phi_{-1-11}\gamma_{\ell_1-1,\ell_2-1,\kappa-1}+\phi_{101}\gamma_{\ell_1-1,\ell_2,\kappa-1}+\phi_{0-11}\gamma_{\ell_1,\ell_2-1,\kappa-1}$$
$$+\phi_{001}\gamma_{\ell_1,\ell_2,\kappa-1}-\theta_{-1-11}\gamma_{za}(\ell_1-1,\ell_2-1,\kappa-1)-\theta_{-101}\gamma_{za}(\ell_1-1,\ell_2,\kappa-1)$$
$$-\theta_{0-11}\gamma_{za}(\ell_1,\ell_2-1,\kappa-1)-\theta_{001}\gamma_{za}(\ell_1,\ell_2,\kappa-1)+\gamma_{za}(\ell_1,\ell_2,\kappa)$$
$$= \phi_1\gamma_{\ell_1,\ell_2,\kappa-1}+\phi_2\gamma_{\ell_1-1,\ell_2,\kappa-1}+ \phi_3\gamma_{\ell_1,\ell_2-1,\kappa-1)}+\phi_4\gamma_{\ell_1-1,\ell_2-1,\kappa-1}$$
$$-\theta_1\gamma_{za}(\ell_1,\ell_2,\kappa-1)-\theta_2\gamma_{za}(\ell_1-1,\ell_2,\kappa-1)-\theta_3\gamma_{za}(\ell_1,\ell_2-1,\kappa-1)$$
$$-\theta_4\gamma_{za}(\ell_1-1,\ell_2-1,\kappa-1)+\gamma_{za}(\ell_1,\ell_2,\kappa)$$

The variance for this ARMA 2 dim model of first order in time and in each space dimension is:

$$(7.3)\quad \gamma_{000} = \phi_1\gamma_{00-1} +\phi_2\gamma_{-10-1} + \phi_3\gamma_{0-1-1} + \phi_4\gamma_{-1-1-1} - \theta_1\gamma_{za}(00-1)$$
$$- \theta_2\gamma_{za}(-101) - \theta_3\gamma_{za}(0-1-1) - \theta_4\gamma_{za}(-1-1-1) + \gamma_{za}(000)$$

(7.4) From equation (2.2) the autocovariance function for $\kappa \geq 2$, $\ell_1 \geq 2$, or $\ell_2 \geq 2$ only depends on the autoregressive coefficients and is given as:

$$\gamma_{\ell_1,\ell_2,\kappa} = \phi_1\gamma_{\ell_1,\ell_2,\kappa-1} + \phi_2\gamma_{\ell_1-1,\ell_2,\kappa-1} + \phi_3\gamma_{\ell_1,\ell_2-1,\kappa-1} + \phi_4\gamma_{\ell_1-1,\ell_2-1,\kappa-1}$$

The remaining autocovariance terms depend on both the autoregressive and the moving average coefficients and are given as:

(7.5)
$$\begin{aligned}
\gamma_{001} = {} & \phi_1\gamma_{000} + \phi_2\gamma_{-100} + \phi_3\gamma_{0-10} + \phi_4\gamma_{-1-10} - \theta_1\gamma_{za}(000 \\
& - \theta_2\gamma_{za}(-100) - \theta_3\gamma_{za}(0-10) - \theta_4\gamma_{za}(-1-10) + \gamma_{za}(001) \\
\gamma_{101} = {} & \phi_1\gamma_{100} + \phi_2\gamma_{000} + \phi_3\gamma_{1-10} + \phi_4\gamma_{0-10} - \theta_1\gamma_{za}(100) \\
& - \theta_2\gamma_{za}(000) - \theta_3\gamma_{za}(1-10) - \theta_4\gamma_{za}(0-10) + \gamma_{za}(101) \\
\gamma_{011} = {} & \phi_1\gamma_{010} + \phi_2\gamma_{-110} + \phi_3\gamma_{000} + \phi_4\gamma_{-100} - \theta_1\gamma_{za}(010) \\
& - \theta_2\gamma_{za}(-110) - \theta_3\gamma_{za}(000) - \theta_4\gamma_{za}(-100) + \gamma_{za}(011) \\
\gamma_{111} = {} & \phi_1\gamma_{110} + \phi_2\gamma_{010} + \phi_3\gamma_{100} + \phi_4\gamma_{000} - \theta_1\gamma_{za}(110) \\
& - \theta_2\gamma_{za}(010) - \theta_3\gamma_{za}(100) - \theta_4\gamma_{za}(000) + \gamma_{za}(111) \\
\gamma_{100} = {} & \phi_1\gamma_{10-1} + \phi_2\gamma_{00-1} + \phi_3\gamma_{1-10} + \phi_4\gamma_{0-1-1} - \theta_1\gamma_{za}(10-1) \\
& - \theta_2\gamma_{za}(00-1) - \theta_3\gamma_{za}(1-10) - \theta_4\gamma_{za}(0-1-1) + \gamma_{za}(100) \\
\gamma_{010} = {} & \phi_1\gamma_{01-1} + \phi_2\gamma_{-11-1} + \phi_3\gamma_{00-1} + \phi_4\gamma_{-10-1} - \theta_1\gamma_{za}(01-1) \\
& - \theta_2\gamma_{za}(-11-1) - \theta_3\gamma_{za}(00-1) - \theta_4\gamma_{za}(-10-1) + \gamma_{za}(010) \\
\gamma_{110} = {} & \phi_1\gamma_{11-1} + \phi_2\gamma_{01-1} + \phi_3\gamma_{10-1} + \phi_4\gamma_{00-1} + \theta_1\gamma_{za}(11-1) \\
& - \theta_2\gamma_{za}(01-1) - \theta_3\gamma_{za}(10-1) - \theta_4\gamma_{za}(00-1) + \gamma_{za}(110) \\
\gamma_{1-10} = {} & \phi_1\gamma_{1-1-1} + \phi_2\gamma_{0-1-1} + \phi_3\gamma_{1-2-1} + \phi_4\gamma_{0-2-1} - \theta_1\gamma_{za}(1-1-1) \\
& - \theta_2\gamma_{za}(0-1-1) - \theta_3\gamma_{za}(1-2-1) - \theta_4\gamma_{za}(0-2-1) + \gamma_{za}(1-10)
\end{aligned}$$

Since this ARMA model is of temporal and spatial order 1, for both the moving average and the autoregressive portions, the only nonzero cross covariance terms are:

$$\gamma_{za}(000) = \sigma_a^2 \quad \gamma_{za}(00-1) = [\phi_1-\theta_1]\sigma_a^2 \quad \gamma_{za}(-10-1) = [\phi_2-\theta_2]\sigma_a^2$$
$$\gamma_{za}(0-1-1) = [\phi_3-\theta_3]\sigma_a^2 \quad \gamma_{za}(-1-1-1) = [\phi_4-\theta_4]\sigma_a^2$$

Substituting these cross covariance terms in equations (7.3) and (7.5) yields the following:

$$\gamma_{000} = \phi_1\gamma_{00-1} + \phi_2\gamma_{-10-1} + \phi_3\gamma_{0-1-1} + \phi_4\gamma_{-1-1-1} - \theta_1(\phi_1-\theta_1)\sigma_a^2$$
$$-\theta_2(\phi_2-\theta_2)\sigma_a^2 - \theta_3(\phi_3-\theta_3)\sigma_a^2 - \theta_4(\phi_4-\theta_4)\sigma_a^2 + \sigma_a^2$$

$$\gamma_{001} = \phi_1\gamma_{000} + \phi_2\gamma_{-100} + \phi_3\gamma_{0-10} + \phi_4\gamma_{-1-10} - \theta_1\sigma_a^2$$

$$\gamma_{101} = \phi_1\gamma_{100} + \phi_2\gamma_{000} + \phi_3\gamma_{1-10} + \phi_4\gamma_{0-10} - \theta_2\sigma_a^2$$

$$\gamma_{011} = \phi_1\gamma_{010} + \phi_2\gamma_{-110} + \phi_3\gamma_{000} + \phi_4\gamma_{-100} - \theta_3\sigma_a^2$$

$$\gamma_{111} = \phi_1\gamma_{110} + \phi_2\gamma_{010} + \phi_3\gamma_{100} + \phi_4\gamma_{000} - \theta_4\sigma_a^2$$

$$\gamma_{100} = \phi_1\gamma_{10-1} + \phi_2\gamma_{00-1} + \phi_3\gamma_{1-10} + \phi_4\gamma_{0-1-1} - \theta_2(\phi_1-\theta_1)\sigma_a^2-\theta_4(\phi_3-\theta_3)\sigma_a^2$$

$$\gamma_{010} = \phi_1\gamma_{01-1} + \phi_2\gamma_{-11-1} + \phi_3\gamma_{00-1} + \phi_4\gamma_{-10-1} - \theta_3(\phi_1-\theta_1)\sigma_a^2 - \theta_4(\phi_2-\theta_2)\sigma_a^2$$

$$\gamma_{110} = \phi_1\gamma_{11-1} + \phi_2\gamma_{01-1} + \phi_3\gamma_{10-1} + \phi_4\gamma_{00-1} - \theta_4(\phi_1-\theta_1)\sigma_a^2$$

$$\gamma_{1-10} = \phi_1\gamma_{1-1-1} + \phi_2\gamma_{0-1-1} + \phi_3\gamma_{1-2-1} + \phi_4\gamma_{0-2-1} - \theta_2(\phi_3-\theta_3)\sigma_a^2$$

Through an iterative procedure, the coefficients for the AR and MA portions may be solved in the terms of the autocovariances. If $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, this ARMA model reduces

to the corresponding 2 dim moving average model of temporal and spatial order 1. The above results for the autocovariance function agree with the results of Voss et al. If the $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, the model reduces to the corresponding 2 dim autoregressive model of first order in time and each of the space dimensions. The autocovariance then agrees with the results of Taneja et al.

The spectrum of this mixed process of dim 2, and first order in time and each space dimension is obtained from equation (3.1) as:

$$p(g,f) = p(g_1,g_{2,f)} = \frac{2\sigma_a^2 \mid \Theta(e^{-i2\pi g}, e^{-i2\pi f}) \mid^2}{\mid \Phi(e^{-i2\pi g}, e^{-i2\pi f}) \mid^2} = \frac{2\sigma_a^2 \mid \Theta(e^{-i2\pi g_1}, e^{-i2\pi g_2}, e^{-i2\pi f}) \mid^2}{\mid \Phi(e^{-i2\pi g_1}, e^{-i2\pi g_2}, e^{-ie\pi f}) \mid^2}$$

$$p(g_1,g_2 f) = \frac{2\sigma_a^2 |1-(\theta_1-\theta_2 e^{-i2\pi g_1} - \theta_3 e^{-i2\pi g_2} - \theta_4 e^{-i2\pi g_1} e^{-i2\pi g_2}) e^{-i2\pi f}|^2}{|1-(\phi_1-\phi_2 e^{-i2\pi g_1} - \phi_3 e^{-i2\pi g_2} - \phi_4 e^{-i2\pi g_1} e^{-i2\pi g_2}) e^{-i2\pi f}|^2},$$

$$0 \leq g_1 \leq \tfrac{1}{2}, \quad 0 \leq g_2 \leq \tfrac{1}{2}, \quad 0 \leq f \leq \tfrac{1}{2}$$

As in the case of the autocovariance function of $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, the spectrum of this 2 dim mixed process reduces to the spectrum of the corresponding 2 dim moving average first order model; and if $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, the spectrum reduces to that of the corresponding 2 dim autoregressive first order model.

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BIBLIOGRAPHY

Akaike, H. (1973). Markovian representation of stochastic processes by canonical variables. SIAM J. of Control, 11.

Anderson, T.W. (1971). The Statistical Analysis of Time Series. New York: John Wiley & Sons, Inc.

Aroian, L.A. (1977). Time Series in m-Dimensions: Definitions, Problems, and Prospects.

Bennett, R.J. (1975). The representation and identification of spatio-temporal systems; an example of population diffusion in North-West England. Institute of British Geographers, Transactions, 66, 73-94.

Box, G.E.P. and Jenkins, G.M. (1976). Time Series Analysis: Forecasting and Control, revised edition. San Francisco: Holden-Day, Inc.

Campbell, D., Schaeffer, D.J., Aroian, L.A. (1977). Pollution in the Chicago Diversion Channel--An Example.

Cliff, A.D., & Ord, J.K. (1975). Model Building and the analysis of spatial pattern in human geography. J. Royal Statist. Soc., Series B, 37, 297-328.

Hannan, E.J. (1970). Multiple Time Series. New York: John Wiley & Sons, Inc.

Jenkins, J.M., & Watts, D.J. (1968). Spectral Analysis and its Applications. San Francisco: Holden-Day, Inc.

Makridakis, S.A. (1956). Survey of time series. International Statist. Rev., 44, 1, 29-70.

Voss, D., Oprian, C. and Aroian, L.A. (1977). Moving average models, time series in m dimensions.

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The paper describes a general linear stochastic model which supposes a time series to be generated by a linear aggregation of random shocks at various temporal and spatial locations.

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